Read the question or problem very carefully

Certain words and phrases (for example, **Find, Prove, Write down**) pop up frequently in questions and problems. These give you an indication of the approach you need to take and the amount of detail you are expected to provide. For example if you are asked to **Write down**, the answer alone will be sufficient. On the other hand, if a question says **Show** you will be expected to provide a step by step argument in detail.

Some of the common words and their meanings are listed here.

Reading the question may also involve checking back on notation, ideas or methods to make sure that you understand the question fully.

- What information have you been given?
- What assumptions have been made?
- Are there any restrictions?

Try this now, by just reading through the following problem (based on a question from MS221). Do not try to work it out now, we will tackle it in stages later!

Problem

Using mathematical induction, prove that if f is the function $f(x) = xe^x$ then for all n in $\mathbb{N}$, the nth derivative of f is given by the formula, $f^{(n)}(x) = (n + x)e^x$.

Hence find the Taylor polynomial of degree n, which is an approximation to this function about $x = 2$. Use your polynomial to write down an approximate value for $2.01e^{2.01}$ correct to 6 decimal places.

Discussion

There are several things to check here – (you can check on the links for further details).

- Can you use the technique of induction confidently?
- Do you understand the notation $\mathbb{N}, e^x, f^{(n)}(x)$?
- Do you know what a Taylor polynomial of degree n looks like?
- Do you understand what is meant by the process words, **Use, Prove, Hence, Find** and **Write down**

Check you understand the problem

Part of understanding the problem is the checking on notation and terminology that you have already done – but you may need to do more than this to get a feel of what the problem involves. There are several things you can try here, depending on the type of problem you are tackling.

If you are trying to establish some general result, it can help to substitute some numbers in for the variables. For example, for the problem above what are $f^{(1)}(x)$ and $f^{(2)}(x)$? Do your answers agree with the result you are trying to establish? Check your answers here.

Problems can often be represented in many ways and drawing a graph or diagram might help you to see the way forward more easily. For example a diagram is often invaluable for highlighting key features of a geometrical problem and a graph can stress aspects of a problem that are difficult to see algebraically.

Can you think of mathematical problems you have worked on, where graphs or diagrams have helped you to solve them?

If you can, break the solution down into small steps

Here the question is broken down into fairly clear steps already and there is also an established technique for tackling a proof by induction. So how do you tackle a proof by induction?

However the question may not have been so clearly specified!

For example, you may have just been asked to:

Find a polynomial approximation for $f(x) = xe^x$ that will enable you to find the value of $2.01e^{2.01}$ correct to 6 decimal places.

Here, you would have had to decide on the intermediate steps, perhaps by asking yourself a series of questions:

‘What methods do I know for finding polynomial approximations?’

‘What techniques do these use?’

‘How many terms will I need in the polynomial?’ and so on.

What mathematics can you use to solve the problem?

Sometimes, as in this case, you will have been given some hints (e.g. Use mathematical induction. . . .) on the approach to take. This may mean that all you have to do is quickly recap on the method and perhaps look at a few examples that use this technique, before being able to launch into the problem quite confidently.

Have a go at the first part of the problem now – can you prove that $f^{(n)}(x) = (n + x)e^x$.

However at other times, it might not be so obvious where to start!

What strategies do you use if you get stuck on a problem like this?

Now try to complete the solution to this problem. You might like to have another look at the work you have completed on Taylor polynomials before checking your solution here.

Checking your work and learning from it

Having solved the problem, it is always worth taking a few moments to look back over your solution, checking through to make sure that you haven't made any mistakes and also that you have included enough detail and written it clearly, so that your reader can follow your ideas easily. Imagine that you have a critical friend who challenges each statement you've made - does your solution answer these challenges or have you left any gaps?

Can you think of a way to check any results by using a different method?

It is also worth thinking about how the problem could be extended and how your solution might be modified to deal with these new situations. This can help to give you a better overall understanding of the topic as well as helping you to prepare for future problems. It helps you to link together the different strands of mathematics.

What extensions can you think of for this problem?

If you get stuck…

Finally, at some point during your course, you are likely to get stuck with a piece of mathematics, either when studying a new piece of work or when you are tackling a problem of your own. This is just part of being a mathematician and you often learn a lot from being stuck and working out a way forward! We have already mentioned a few strategies earlier in this section – here are a few more ideas to try.

Try discussing the problem with someone else – either face-to-face, on the phone or on-line. Sometimes, just trying to explain what the problem is and what you are trying to do will clarify your thoughts sufficiently for you to see a way forward, and if it doesn't they may be able to suggest an alternative way of looking at the problem which could help.

Have a break! You may find that leaving the problem for a while and doing something completely different for a few hours or even days, enables you to come back to the problem with fresh ideas and alternative ways of tackling it.

Try reading more about the subject or finding alternative sources – a different textbook, a maths package or a web resource such as an interactive program may help you to see a different way of approaching the problem. Sometimes just reading on in the unit can help – as you begin to understand more about the topic, gaps from earlier on can sometimes be filled fairly easily!

Have a look back over your previous work and examples – you may just have forgotten a key condition, result or technique.

Contact your tutors – they are there to help!

HAPPY PROBLEM SOLVING!

End of main text. You can now close this page and return to the home page where you can choose to visit a different section.

Note

This is a powerful method which can be used for proving many different mathematical propositions. You may like to refer to your work on a previous course (for example, MS221) or the course you are currently studying, (for example, M203 or M208) or search on the web for further help with this topic. To view an example, please visit the following web site. To return to this note from the web site press BACK on the menu bar.

OK

Scribbles

This is a product of the two functions, x and e^x. Using the product rule,

$$\begin{aligned} f^{(1)}(x) &= 1 \times e^x + x \times e^x \\ &= (1+x)e^x. \end{aligned}$$

This is a product of the two functions, e^x and $(1+x)$. So, using the product rule again,

$$\begin{aligned} f^{(2)}(x) &= 1 \times e^x + (1+x) \times e^x \\ &= (2+x)e^x. \end{aligned}$$

Hopefully by working on these special cases, you will gain a better understanding of the problem and the methods that might be used in the solution of the main problem. For example, here you have used the product rule for differentiation and then simplified the resulting expression by collecting the terms in e^x to get the desired form.

You should, of course, be aware that showing a result is true for specific values does not constitute a proof for the general case.

OK

Thinks

A proof by induction usually involves three main steps:

- Specifying the proposition $p(n)$ and finding a case where the proposition is true, for example $p(1)$.
- Proving that if the proposition is true for an intermediate case, it is also true for the next case. This is often achieved by assuming that the proposition $p(k)$ is true and then by manipulating this result algebraically, to establish that the proposition $p(k+1)$ is also true.
- Deducing that the proposition is true for all values of $n \in \mathbb{N}$, greater than or equal to the value found in step 1.

OK

Scribbles

Let $p(n)$ be the proposition $f^{(n)}(x) = (n+x)e^x$ where $f(x) = xe^x$. First check that $p(1)$ is true.

Find the derivative of f using the Product Rule: $f^{(1)}(x) = e^x + xe^x = (1+x)e^x$.

This agrees with the formula $f^{(n)}(x) = (n+x)e^x$ with $n = 1$, so $p(1)$ is true.

Now, show that if $p(k)$ is true, then $p(k+1)$ is also true.

$f^{(k+1)}(x)$ is the derivative of $f^{(k)}(x)$. So differentiating $f^{(k)}(x)$ gives

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d(f^{(k)}(x))}{dx} \\ &= \frac{d((k+x)e^x)}{dx} \\ &= e^x + (k+x)e^x \text{ by the product rule} \\ &= ((k+1)+x)e^x. \end{aligned}$$

So $p(k+1)$ is also true.

Hence, $p(1)$ is true and also $p(k) \Rightarrow p(k+1)$ is true for $k = 1, 2, 3 \ldots$.

So, we can deduce by mathematical induction that $p(n)$ is true for all $n \in \mathbb{N}$.

OK

Scribbles

The Taylor polynomial $p(x)$ of degree n about a for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In this case, $a = 2$ and $f^{(n)}(x) = (n+x)e^x$.

Hence $f^{(n)}(2) = (n+2)e^2$. So,

$$\begin{aligned} p(x) &= 2e^2 + (1+2)e^2(x-2) + \frac{(2+2)e^2}{2!}(x-2)^2 + \frac{(3+2)e^2}{3!}(x-2)^3 + \ldots + \frac{(n+2)e^2}{n!}(x-2)^n \\ &= 2e^2 + 3e^2(x-2) + 2e^2(x-2)^2 + \frac{5e^2}{6}(x-2)^3 + \ldots + \frac{(n+2)e^2}{n!}(x-2)^n. \end{aligned}$$

So, when $x = 2.01$

$$p(2.01) = 2e^2 + 3e^2 \times 0.01 + 2e^2 \times 0.01^2 + \frac{5e^2}{6} \times 0.01^3 + \ldots + \frac{(n+2)e^2}{n!} \times 0.01^n.$$

Taking the first four terms, $p(2.01) = 15.00126785$ and taking the first five terms, $p(2.01) = 15.00126787$.

Rounding off to 6 decimal places, both these approximations give the same value of 15.001268.

OK

Thinks

Here a quick and easy way to check your answer would be to work out $2.01e^{2.01}$ using a calculator or maths package. You should find that the calculator value (15.00126787) does match the one you found! There are often other ways of checking that your result does seem reasonable, either by using a maths package such as Mathcad, making an estimate or using an alternative approach.

OK

Thinks

You can think of extensions to problems by asking yourself 'What if...?' questions and changing the conditions of the problem. So some ideas here might be:

- What if we changed the function? What sort of functions will this method work for?
- What if we changed the value? If we find the polynomial approximation about 2, for what range of values will the approximation be reasonable? How could we get a feel for this? How could we check it?
- What if we changed the method? Are there other ways of finding polynomial approximations? Could we use an existing Taylor polynomial say for e^x? Which methods converge quickly?

OK

Note

Meanings of common words and phrases used in problems.

Write down... What is...	An answer is all that is required. However, if you show some working, you may get some marks even if your answer is wrong.
Determine... Find... Calculate... Derive...	Justification for your answer is required so you will need to explain your working step by step.
Using...	You are told which method to use – using a different approach will lose marks.
Show... Verify... Prove...	The answer is given to you. All marks are awarded for a convincing argument.
Hence... Hence, or otherwise... Deduce...	Use the result you have just established to solve the next part of the problem. Note that 'or otherwise' means that you can use an alternative method if you wish, but it's often easier to try using your previous work first!

OK

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OK

Note

These are:

$\mathbb{N}$ is the set of natural numbers: 1, 2, 3 . . . .

e^x is the exponential function. It is sometimes written $\exp(x)$.

$f^{(n)}(x)$ is the nth derivative of f with respect to x. For example,

$$f^{(1)}(x) = \frac{d(f(x))}{dx} \text{ and } f^{(2)}(x) = \frac{d^2(f(x))}{dx^2}.$$

OK