Taylor polynomials of degree n about a

In an earlier section we considered polynomials about zero. However it is possible to generalize the derivation of the formula for a Taylor polynomial about zero that you saw earlier to obtain a formula for a polynomial that approximates a suitable function f close to any chosen point a in its domain. The required formula is given below

Here suitable means that f must be differentiable at a the required number of times.

Taylor polynomials about a
Let f be a function that is n-times differentiable at a. The **Taylor polynomial** of degree n about a for f is

$$\begin{aligned} p(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &\quad + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

When $a = 0$, this reduces to the formula for Taylor polynomials about 0.

As in the case $a = 0$ the formula shows that any Taylor polynomial about a point a for a function f can be obtained from a Taylor polynomial of lower degree about a for f by adding the appropriate further terms. Also, again as in the case $a = 0$ it is possible for the Taylor polynomial of degree n about a for f to be of degree less than n; this happens when $f^{(n)} = 0$.

If p is a Taylor polynomial for a function f about a point a in its domain, and x is a point close to a, then $p(x)$ is an approximation for $f(x)$. Usually the greater the degree of the Taylor polynomial, and the closer x is to a, the more accurate the approximation.

Example Find the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$.

Solution The first four derivatives of the function $f(x) = \ln x$ are as follows:

The product 3×2 has been simplified to 3! rather than 6 here to highlight the patterns emerging.

$$f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f^{(3)}(x) = \frac{2}{x^3}, f^{(4)}(x) = -\frac{3\times 2}{x^4} = -\frac{3!}{x^4}.$$

Evaluating f and its first four derivatives at 1 gives:

$$f(1) = 0, f'(1) = 1, f''(1) = -1, f^{(3)}(1) = 2, f^{(4)}(1) = -3!.$$

So by the formula above, the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$ is

$$0 + 1 \times (x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4;$$

that is,

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

Notice that we do not multiply out the brackets in this expression. We usually leave a Taylor polynomial about a point a as a sum of terms each of which is the product of a constant and a power of $x - a$.

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