Quadratic Taylor polynomials about 0

We now look at approximating functions by quadratic functions.

You might expect this to give greater accuracy than approximating a function by a linear function, and this is usually the case. For simplicity in this subsection we consider approximations about zero only. However in section two we study approximations by polynomials of any degree n and there you will see how to find approximations about a general point a.

Suppose, that f is a differentiable function whose domain contains 0. In an earlier section you saw how to approximate f close to 0 by a function p of the form $p(x) = a_0 + a_1 x$ which was chosen to be the function whose graph is the tangent to the graph of f at 0. In other words, p was chosen to be the linear function that satisfies the following two conditions:

(i) the values of the function and the approximation are equal at 0; that is, $p(0) = f(0)$;

(ii) the values of the first derivatives of the function and the approximation are equal at 0; that is, $p'(0) = f'(0)$.

Usually this linear approximation is good for values of x close to zero, but its accuracy decreases as x moves away from zero.

Suppose that we now wish instead to consider approximating f close to 0 by a function p of the form

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

It seems sensible to require this new function p to satisfy conditions (i) and (ii) above. As p now has 3 coefficients, we can also impose a third condition, and a natural one to choose is

(iii) the values of the second derivatives of the function and the approximation are equal at 0; that is, $p''(0) = f''(0)$.

We can impose this condition provided that $f''(0)$ exists; that is provided that f is twice differentiable at 0.

Many real functions can be differentiated as often as we wish at all points in their domains. These include polynomial, rational, trigonometric, exponential and logarithmic functions, and combinations and compositions of these.

You have seen that conditions (i) and (ii) ensure that the graphs of the function and the approximation both pass through the same point $(0, f(0))$ and have the same gradient at that point. Condition (iii) ensures that the function and approximation also have the same rate of change of gradient at 0. (Roughly speaking, this means that their graphs have the same curvature at the point $(0, f(0))$).

The polynomial p that satisfies conditions (i),(ii) and (iii) is called the **quadratic Taylor polynomial** about 0 for f. For any point x close to 0, the value of $p(x)$ is an approximation for $f(x)$.

In some cases, the polynomial $p(x) = a_0 + a_1x + a_2x^2$ that satisfies conditions (i), (ii) and (iii) has $a_2 = 0$ and so is *not* a quadratic polynomial but has degree 1 or less. If this happens, then we still refer to the approximating polynomial as the *quadratic* Taylor polynomial about 0 for f. This means that a quadratic Taylor polynomial is not necessarily a quadratic polynomial!

End of main text. You can close this file and move on to the second activity.

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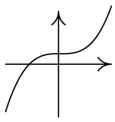
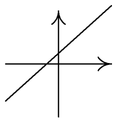
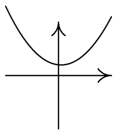
Function $f(x)$



Linear $ax + b$



Quadratic $ax^2 + bx + c$



Why would a quadratic work better? Well a curve should fit better than a line (at least near 0). Ah! algebraically there is an extra term.

OK

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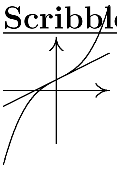

Not bad but goes a bit awry as you move away from 0.

OK